670

VOLUME 81 SEPARATE No. 670

PROCEEDINGS

AMERICAN SOCIETY
OF
CIVIL ENGINEERS

APRIL, 1955

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

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ENGINEERING MECHANICS
DIVISION

{Discussion open until August 1, 1955}

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

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This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

VIRTUAL MASS AND ACCELERATION IN FLUIDS

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SYNOPSIS

The "effective inertia" of a body is greater in air than in a vacuum and is much greater still in liquid. This "effective inertia," which is greater than the "quantity of matter" or mass (ratio of weight to the acceleration due to gravity) of a body, is called virtual mass. The excess of virtual mass over the "quantity of matter" is the added mass due to the surrounding fluid.

Measurements were made on objects from two to twenty inches in largest dimension as they were immersed in tap water and accelerated in oscillatory motion. The added masses, which depend on the size and shape of the body, the direction of acceleration, and the density of the fluid, were determined from a mass-frequency relationship for the supporting beam. The experimental results are reasonable and consistent, and they agree with previous analytical studies of potential flow.

INTRODUCTION

In 1779 Du Buat (1) published the results of his observations on spherical pendulum bobs of lead, glass, and wood vibrating in air and in water. He noticed that a simple buoyancy correction for the submerged sphere was not sufficient. In addition, the fluid had the same effect as increasing the mass of the sphere by an amount equal to approximately one-half the mass of the fluid that was displaced.

Since the time of Du Buat the added mass due to submergence in a fluid has been the subject of many analytical and experimental investigations(2). Determination of added mass by experiment often has been inconsistent and unreasonable. Measured values for added mass have usually been larger than can be explained by theory and analysis. Luneau(3) reported on tests of circular disks where he found added-masses from 3 to 9 times the theoretical values. In 1951 Iverson and Balent(4) concluded that added mass is a variable which depends on the state of motion and is not a constant as had been shown by studies of potential flow.

In the present study a new method for determining the added mass of submerged bodies was developed. The method is simple yet accurate, and permits the easy measurement of added mass for bodies of any shape. The measured values are in substantial agreement with analytical values obtained from studies of potential flow. Thus, the practical value of added masses obtained from analytical studies is confirmed. In addition, for odd-shaped bodies a reliable experimental method is now available to determine the added masses which cannot be obtained by analysis alone.

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Theory

If a body of mass m moves in rectilinear motion at a velocity v its kinetic energy T_B is

$$T_B = \frac{1}{2} mv^2$$

If a body moves at a velocity v in an ideal, incompressible fluid, of infinite extent, at rest of infinity, and in irrotational motion, the total kinetic energy of the fluid T_F is

$$T_F = \frac{1}{2} cv^2$$

where c is a constant that depends only on the density of the fluid, the size and shape of the body, and the direction of motion(5). The total kinetic energy T of the system (body and fluid) is

$$T = T_F + T_B = \frac{1}{2} (m+c) v^2$$

When the body is accelerated, the rate of change in the total kinetic energy with respect to time t is

$$\frac{dT}{dt} = (m + c)v \frac{dv}{dt}$$

But

$$\frac{dT}{dt} = F\ v$$

where F is the resultant of the external forces that act on the body to cause the acceleration, $\frac{dv}{dt}$. Hence,

$$F\ v = (m + c)v \frac{dv}{dt}$$

and

$$F = (m + c) \frac{dv}{dt}$$

Thus, the fluid has the effect of increasing the inertia or mass of the body from m to (m + c). The term c is called the added mass and (m + c) the virtual mass. A more rigorous explanation of this phenomenon can be found in textbooks on hydrodynamics(6,7). Energy rather than momentum expressions have been used in the above explanation since in the analysis of potential flow the energy integrals usually converge and momentum integrals do not.

The virtual mass is the inertia coefficient or ratio of applied force to acceleration in fluid media. The actual mass of a body is the inertia coefficient that applies to movement in a vacuum. Thus, the quantity that is of practical importance in most dynamic problems is the virtual mass; yet the difference between actual and virtual mass is frequently overlooked or assumed to be of only academic interest. For movement in low-density fluids such as air, the difference may not be significant; but in fluids such as water

the virtual mass may be many times the actual mass. Consequently, the difference between the two may be really important.

Practical Importance

Westergaard(8) investigated the effect of virtual mass on dams during earthquakes. He found the inertia-pressures to be neither excessively large nor negligible.

Dunn(9) and Ackerman(10) described a unique method of placing Chute-a-Caron cofferdam in the Saguenay River. The dam, 92 feet long, 45 feet thick, and 42 feet high at the center, was built on one end on a pier and toppled into place in the river. The river was 28 feet deep and flowed at a velocity of 20 feet per second. Although the 11,000 tons of concrete in the dam fell an average distance of 86 feet and came to rest on bed rock, only a few hairline cracks developed. The water "cushioned" the fall so that the final impact was the equivalent of only a 4-inch drop through air. The potential energy of the falling dam was largely changed into kinetic energy of the water. Velocities of water were estimated to be as high as 600 feet per second.

The added mass effect has been considered on such diverse problems as roll, pitch and vibration of ships; accelerations of submarines, ships and blimps; entrance of projectiles and sea plane floats into water; sediment movement and wave action; tone of bells and strings; and the vibration of plates and structures.

Whenever solids in contact with fluids accelerate, virtual mass is a factor that should not be overlooked.

Experimental Method

A sketch of the apparatus is shown in Fig. 1. The test bodies were immersed in a tank of tap water 5 feet 4 inches in diameter and 2 feet 9 inches deep. The centroids were placed 1 foot 3 inches under the surface in the center of the tank. Supporting rods were rigidly attached to a beam 3/8 inch deep, 1/8 inch wide, and 24 inches long. The ends of the beam were simply supported on pivot bearings just above the surface of the water.

Before testing the bodies, the relationship between the natural fundamental frequency for vertical vibration of the beam and the mass in air attached to the center of the beam was determined by experiment.

The frequency was measured by attaching a small magnet to the beam. When vibrated, the moving magnet produced a potential in a surrounding coil that was independently supported. An oscillograph recorded the varying potential in the coil. Frequency was obtained by averaging the time for 30 to 50 cycles on the oscillograph record.

A test body was then submerged in water, attached to the center of the beam by a rod, and the beam was vibrated at its new natural fundamental frequency. The frequency of the body submerged in water was equal to the frequency of a body of larger mass attached to the beam and vibrating freely in air. The difference between the mass of the submerged body and the attached mass in air that gives the same frequency is the added mass effect of the fluid.

The foregoing statement is true only if the following assumptions are made:

1. The weight of air displaced by the weights attached to the beam is negligible compared to the weight of water displaced by the submerged

bodies. Since the ratio of the densities of air and water is about 1 to 800 and since the attached masses are relatively very compact in shape and size, error due to the added mass in air could not be more than one part in 2000.

2. The effect of the viscosity of the water is assumed to be negligible. The validity of this assumption is supported by two experimental observations: (a) The oscillograph trace which represents the vertical velocity of the beam is as nearly sinusoidal as can be measured. (b) The damping is so slight that the amplitude of the vibration is reduced by only one-half in 20 to 50 cycles—a logarithmic damping decrement of about 2 percent per cycle.

Lamb[11] analyzed the effect of viscosity on the inertia resistance of a sphere oscillating in simple harmonic motion. The viscosity changes the inertia coefficient from

$$\frac{1}{2} \text{ to } \frac{1}{2} + \frac{9}{4a}\left(\frac{\nu}{\pi f}\right)^{\frac{1}{2}}$$

where a = radius of the sphere, f = frequency of the vibration, and ν = kinematic viscosity of the fluid. For typical test conditions of a sphere having a radius of 0.17 feet vibrating in water (ν = 0.00001 ft²/sec) at 20 cycles per second, the change in inertia resistance due to viscosity is

$$\frac{9}{4(0.17)}\left(\frac{0.00001}{\pi\, 20}\right)^{\frac{1}{2}} = 0.0054$$

or about one percent. Thus, viscosity would not seriously affect the experimental values for added mass.

Results and Analysis

Spheres.—Fig. 2 shows that the measured added mass due to the fluid is 0.51 times the mass of fluid displaced. This agrees well with theoretical analyses [7] that show the added mass to be one-half the displaced mass.

Cubes.—Fig. 2 also shows that the measured added mass is 0.67 times the displaced mass for cubes accelerated either broadside-on or edge-on.

The authors know of no theoretical solution to this problem. However, as explained by Polya [12] and Szego [13], for a given body the sum of the translational added masses along three mutually perpendicular axes is an invariant. Thus, if M_b is the added mass for a cube moving broadside-on and if M_e is the added mass for edge-on movement, mutually perpendicular axes can be selected such that

$$3\,M_b = 2\,M_e + M_b$$

Therefore,

$$M_b = M_e$$

Thus, theory indicates that the added masses of a cube for edge-on and broadside-on motion should be identical as was shown by experiment.

The ratio of added mass to displaced mass for a cube was 0.67—one-third greater than the corresponding ratio, 0.51, for a sphere. Szego [13] has shown that a sphere has less added mass than any other nearly-spherical solid of equal volume. Hence, theory would indicate that the ratio for cubes should be larger than the ratio for spheres as was shown by the authors' experiments.

Circular Cylinders.—Fig. 3 shows the relationship between the ratio of length to diameter and the ratio of added mass to the displaced mass for square-ended circular cylinders of wood about 2 inches in diameter. The results are plotted on "semi-hyperbolic" paper for which "Y-distances" from the origin vary directly as the "variable y," and "X-distances" from the origin vary inversely as the "variable x." Experimental data from the authors' tests of circular cylinders, shown by the solid circles, define a line which intersects the axis of ordinates at 1.00. This means that, experimentally, for rigid circular cylinders whose ratio of length to diameter is "very large," the added mass is 1.00 times the displaced-cylinder mass—exactly the figure indicated by mathematical analysis.

Rectangular Plates.—Rectangular plates 2 to 3 inches wide and 2 to 8 inches long of 16-gage sheet steel were accelerated broadside-on. The results of these tests are shown in Fig. 3 as open circles. As the length to width ratio for the plates becomes large, the ratio of added mass to displaced-cylinder mass approaches 1.05. The displaced-cylinder mass is by definition the mass of fluid displaced by a circular cylinder with flat, square ends having a diameter equal to the width of the plate. The plate and cylinder lengths are the same.

Riabouchinsky[14] analyzed potential flow about a cylinder of infinite length having a rectangular cross-section moving broadside-on. He found the added mass, c, per unit length in a fluid of unit density to be

$$c = \frac{\pi}{4} w^2 \left\{ \frac{\sin^2\alpha}{\left[E - (\cos^2\alpha)K\right]^2} - \frac{4}{\pi}\frac{d}{w} \right\}$$

where α is a parameter such that

$$\frac{d}{w} = \frac{E' - (\sin^2\alpha)K'}{E - (\cos^2\alpha)K}$$

The broadside width is w, and the thickness in the direction of motion is d. The terms K and E are complete elliptic integrals of the first and second kind and $\cos\alpha$, $\sin\alpha$ are their respective moduli. The terms E' and K' are the corresponding functions of the complementary angle $(90° - \alpha)$.

When $\alpha = 80°$, Riabouchinsky found that

$$\frac{d}{w} = 0.025 \text{ and } \frac{c}{\frac{\pi}{4}w^2} = 1.05$$

The quantity

$$\frac{\pi}{4}w^2$$

is the displaced-cylinder mass in a fluid of unit density.

In the experiments $d = 0.0625$ inches and the mean width $w = 2.50$ inches. Thus,

$$\frac{d}{w} = \frac{0.0625}{2.50} = 0.025$$

and by analysis the ratio of added mass to displaced-cylinder mass for such plates of infinite length should be 1.05 as shown by the authors' experiments.

Rectangular Parallelepipeds.—Fig. 4 shows the relationship between the ratio of added mass to displaced-cube mass and the ratio of thickness in the direction of motion to the edge width perpendicular to motion for rectangular parallelepipeds having a square side moving broadside-on. The displaced-cube mass is defined as the mass of fluid displaced by a cube having sides the same as the square sides of the parallelepiped. The thickness of the test bodies varied from 0.06 to 6 inches, and the width or side-length of the square side varied from 2 to 4 inches. The bodies were of steel, plastic, and wood.

CONCLUSIONS

1. A simple experimental method is presented for measuring the added mass of bodies accelerated in liquids.
2. The measured added mass agrees well with that obtained from an analysis of potential flow for some simple shapes that have been analysed. A comparison of results follows:

Body	Added Mass (given as a fraction of the mass of fluid displaced by body) By Analysis	By Experiment
Sphere	0.50	0.51
Long circular cylinder accelerated transversely	1.00	1.00 (asymptote)
Long, thin rectangular cylinder accelerated broadside-on	1.05*	1.05* (asymptote)

*Referred to the displaced mass of a circular cylinder inscribed in the broadside width.

3. For those shapes that have not been analysed the added mass has been measured as follows:

Body	Range	Added Mass
Cube accelerated parallel to a side	-----	0.67 displaced mass
Circular cylinder accelerated transversely	$1.2 \leq \frac{\text{length}}{\text{diameter}} \leq \infty$	as shown in Fig. 3
Thin rectangular cylinder accelerated broadside-on	$0 \leq \frac{\text{length}}{\text{diameter}} \leq \infty$	as shown in Fig. 3
Rectangular parallelepipeds with square side accelerated broad-side-on	$0.016 \leq \frac{\text{thickness}}{\text{width}} \leq 2$	as shown in Fig. 4

ACKNOWLEDGMENTS

The study from which this paper is drawn was supported by financial aid from a Gerard Swope Fellowship awarded by the General Electric Educational Fund.

The Experiments were performed in the Civil Engineering Laboratories at Carnegie Institute of Technology.

Dr. C. F. Peck, Jr. and Mr. T. E. Doyle, Jr., contributed significantly to the development of experimental methods of producing and recording vibrations.

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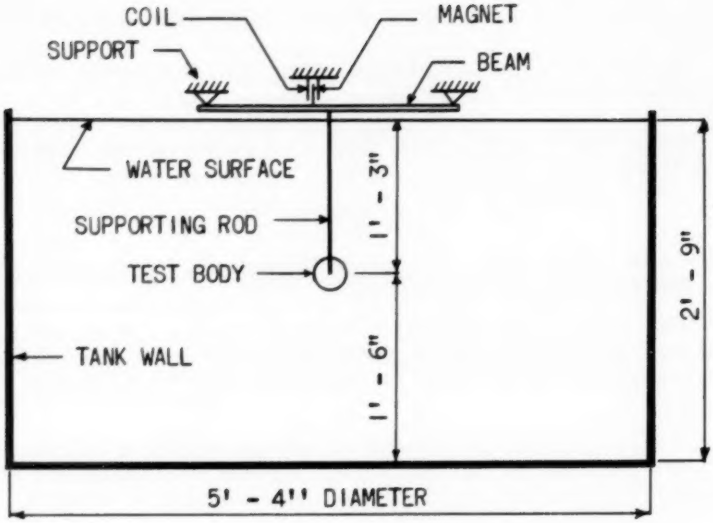

FIGURE 1. APPARATUS FOR MEASURING ADDED MASS.

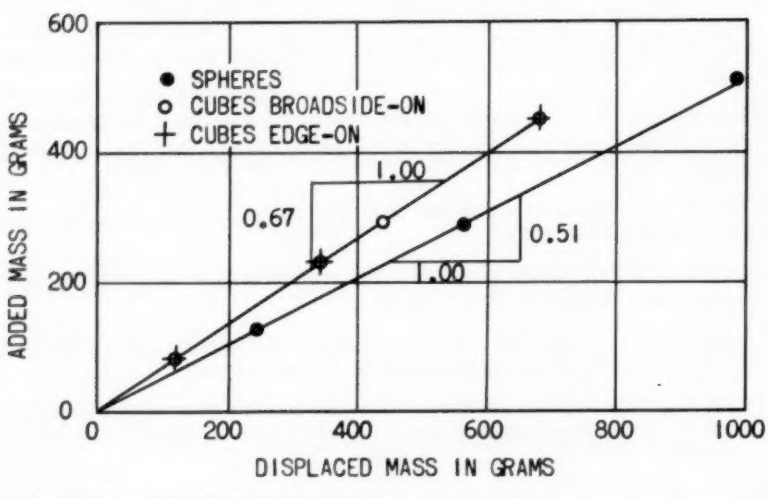

FIGURE 2. SPHERES AND CUBES.

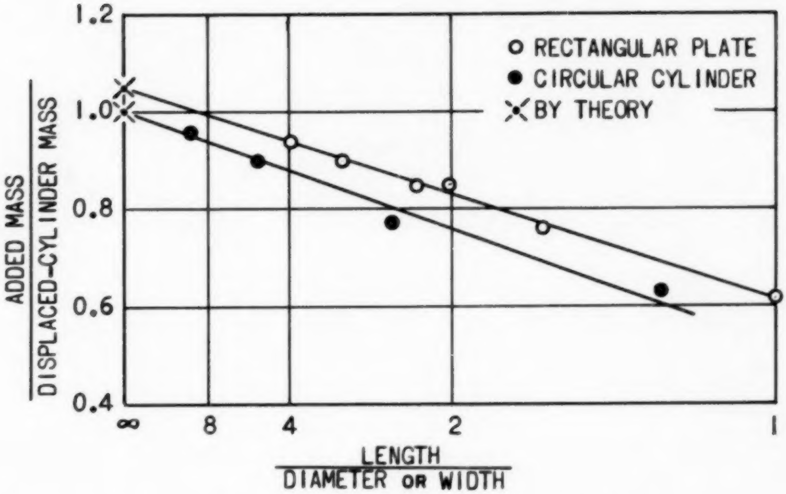

FIGURE 3. CIRCULAR CYLINDERS AND RECTANGULAR PLATES.

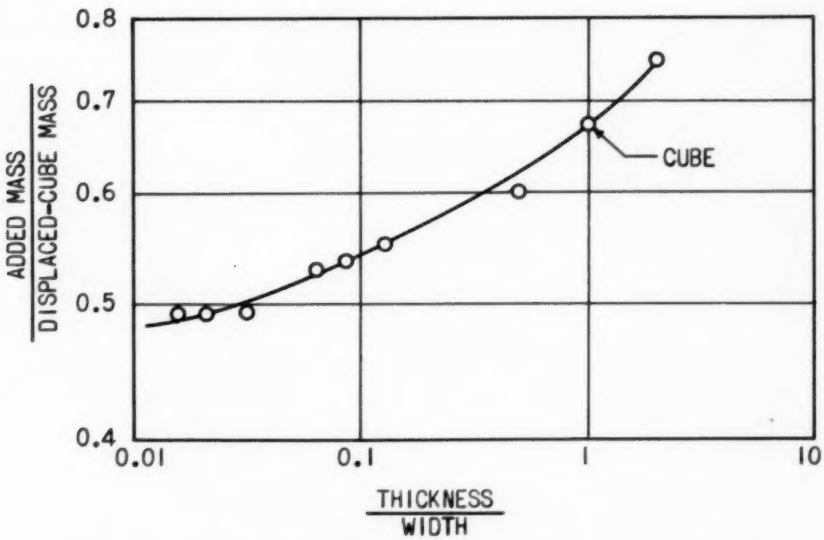

FIGURE 4. RECTANGULAR PARALLELEPIPEDS WITH SQUARE SIDE MOVING BROADSIDE-ON.